

# BRAIDINGS ON THE CATEGORY OF BIMODULES, AZUMAYA ALGEBRAS AND EPIMORPHISMS OF RINGS

A. L. AGORE, S. CAENEPEEL, AND G. MILITARU

**ABSTRACT.** Let  $A$  be an algebra over a commutative ring  $k$ . We prove that braidings on the category of  $A$ -bimodules are in bijective correspondence to canonical  $R$ -matrices, these are elements in  $A \otimes A \otimes A$  satisfying certain axioms. We show that all braidings are symmetries. If  $A$  is commutative, then there exists a braiding on  ${}_A\mathcal{M}_A$  if and only if  $k \rightarrow A$  is an epimorphism in the category of rings, and then the corresponding  $R$ -matrix is trivial. If the invariants functor  $G = (-)^A : {}_A\mathcal{M}_A \rightarrow \mathcal{M}_k$  is separable, then  $A$  admits a canonical  $R$ -matrix; in particular, any Azumaya algebra admits a canonical  $R$ -matrix. Working over a field, we find a remarkable new characterization of central simple algebras: these are precisely the finite dimensional algebras that admit a canonical  $R$ -matrix. Canonical  $R$ -matrices give rise to a new class of examples of simultaneous solutions for the quantum Yang-Baxter equation and the braid equation.

## INTRODUCTION

Braided monoidal categories play a key role in several areas of mathematics like quantum groups, noncommutative geometry, knot theory, quantum field theory and 3-manifolds. It is well-known that the category  ${}_A\mathcal{M}_A$  of bimodules over an algebra  $A$  over a commutative ring  $k$  is monoidal. The aim of this paper is to give an answer to the following natural question: given an algebra  $A$ , describe all braidings on  ${}_A\mathcal{M}_A$ . Besides the purely categorical significance this problem is also relevant in noncommutative geometry where braidings on  ${}_A\mathcal{M}_A$  are used to develop the theory of wedge products of differential forms or connections on bimodules. The question is not as obvious as it seems: a first attempt might be to use the switch map to define the braiding, but this is not well-defined, even in the case when  $A$  is a commutative algebra. However, there are non-trivial examples of braidings on the category of bimodules. For example, let  $A = \mathcal{M}_n(k)$  be a matrix algebra; then  $c_{M,N} : M \otimes_A N \rightarrow N \otimes_A M$  given by the formula

$$c_{M,N}(m \otimes_A n) = \sum_{i,j,t=1}^n e_{ij} n e_{ti} \otimes_A m e_{jt}$$

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is a braiding on the category of  $A$ -bimodules (see Example 2.9 for full detail). A first general result is Theorem 2.1, stating that braidings on the category of  $A$ -bimodules are in bijective correspondence with canonical R-matrices, these are invertible elements  $R$  in the threefold tensor product  $A \otimes A \otimes A$ , satisfying a list of axioms. In this situation, we will say that  $(A, R)$  is an algebra with a canonical R-matrix. Actually, this result is inspired by a classical result of Hopf algebras: braidings on the category of (left) modules over a bialgebra  $H$  are in one-to-one correspondence with quasitriangular structures on  $H$ , these are elements  $R$  in the two-fold tensor product  $H \otimes H$  satisfying certain properties. We refer to [12, Theorem 10.4.2] for detail. The next step is to reduce the list of axioms to two equations, a centralizing condition and a normalizing condition, and then we can prove in Theorem 2.2 that all braidings on a category of bimodules are symmetries. In the situation where  $A$  is commutative, we have a complete classification:  $A$  admits a canonical R-matrix  $R$  if and only if  $k \rightarrow A$  is an epimorphism in the category of rings, and then  $R$  is trivial, see Proposition 2.3.

The invariants functor  $G = (-)^A : {}_A\mathcal{M}_A \rightarrow \mathcal{M}_k$  has a left adjoint  $F = A \otimes -$ . We prove that  $G$  is a separable functor [13, 14] if and only if  $G$  is fully faithful and this implies that  $A$  admits a canonical R-matrix. The converse property also holds if  $A$  is free as a  $k$ -module, and then the braiding on the category of  $A$ -bimodules is unique, cf. Theorem 2.6.

Azumaya algebras were introduced in [1] under the name central separable algebras; a more restrictive class was considered earlier by Azumaya in [2]. Azumaya algebras are the proper generalization of central simple algebras to commutative rings. The Brauer group consists of the set of Morita equivalence classes of Azumaya algebras. There exists a large literature on Azumaya algebras and the Brauer group, see for example the reference list in [4].  $A$  is an Azumaya algebra if and only if  $G$  is an equivalence of categories, and then  $G$  is separable. Therefore the category of bimodules over an Azumaya algebra is braided monoidal, that is any Azumaya algebra admits a canonical R-matrix.  $R$  can be described explicitly in the cases where  $A$  is a matrix algebra or a quaternion algebra, see Examples 2.9 and 2.10. Not every algebra with a canonical R-matrix is Azumaya; for example  $\mathbb{Q}$  is not a  $\mathbb{Z}$ -Azumaya algebra, but  $1 \otimes 1 \otimes 1$  is a canonical R-matrix, since  $\mathbb{Z} \rightarrow \mathbb{Q}$  is an epimorphism of rings. Thus algebras with a canonical R-matrix can be viewed as generalizations of Azumaya algebras.

Applying Theorem 2.6 to finite dimensional algebras over fields, we obtain a new characterization of central simple algebras, namely central simple algebras are the finite dimensional algebras admitting a canonical R-matrix. As a final application, we construct a simultaneous solution of the quantum Yang-Baxter equation and the braid equation from any canonical R-matrix, see Theorem 2.13.

## 1. PRELIMINARY

**Azumaya algebras.** Let  $k$  be a commutative ring and  $A$  a  $k$ -algebra. Unadorned  $\otimes$  means  $\otimes_k$ .  ${}_A\mathcal{M}_A$  is the  $k$ -linear category of  $A$ -bimodules. It is well-known that we have a pair of adjoint functors  $(F, G)$  between the category of  $k$ -modules  $\mathcal{M}_k$  and the category of  $A$ -bimodules  ${}_A\mathcal{M}_A$ . For a  $k$ -module  $N$ ,  $F(N) = A \otimes N$ , with  $A$ -bimodule structure  $a(b \otimes n)c = abc \otimes n$ , for all  $a, b, c \in A$  and  $n \in N$ . For an  $A$ -bimodule  $M$ ,

$G(M) = M^A = \{m \in M \mid am = ma, \forall a \in A\} \cong {}_A\text{Hom}_A(A, M)$ . The unit  $\eta$  and the counit  $\varepsilon$  of the adjoint pair  $(F, G)$  are given by the formulas

$$\begin{aligned} \eta_N : N &\rightarrow (A \otimes N)^A & ; & \quad \eta_N(n) = 1 \otimes n; \\ \varepsilon_M : A \otimes M^A &\rightarrow M & ; & \quad \varepsilon_M(a \otimes m) = am = ma \end{aligned}$$

for all  $n \in N$ ,  $a \in A$  and  $m \in M^A$ . Recall that  $A$  is an Azumaya algebra if  $A$  is faithfully projective as a  $k$ -module, that is,  $A$  is finitely generated, projective and faithful, and the algebra map

$$(1) \quad F : A^e = A \otimes A^{\text{op}} \rightarrow \text{End}_k(A), \quad F(a \otimes b)(x) = axb$$

is an isomorphism. Azumaya algebras can be characterized in several ways; perhaps the most natural characterization is the following:  $A$  is an Azumaya algebra if and only if the adjoint pair  $(F, G)$  is a pair of inverse equivalences, see [11, Theorem III.5.1]. Another characterization is that  $A$  is central and separable as a  $k$ -algebra.

**Separable functors.** Recall from [13] that a covariant functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called separable if the natural transformation

$$\mathcal{F} : \text{Hom}_{\mathcal{C}}(\bullet, \bullet) \rightarrow \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) ; \quad \mathcal{F}_{C, C'}(f) = F(f)$$

splits, that is, there is a natural transformation

$$\mathcal{P} : \text{Hom}_{\mathcal{D}}(F(\bullet), F(\bullet)) \rightarrow \text{Hom}_{\mathcal{C}}(\bullet, \bullet)$$

such that  $\mathcal{P} \circ \mathcal{F}$  is the identity natural transformation. Rafael's Theorem [14] states that the left adjoint  $F$  in an adjoint pair of functors  $(F, G)$  is separable if and only if the unit of the adjunction  $\eta : 1_{\mathcal{C}} \rightarrow GF$  splits; the right adjoint  $G$  is separable if and only if the counit  $\varepsilon : FG \rightarrow 1_{\mathcal{D}}$  cosplits, that is, there exists a natural transformation  $\zeta : 1_{\mathcal{D}} \rightarrow FG$  such that  $\varepsilon \circ \zeta$  is the identity natural transformation. A detailed study of separable functors can be found in [5].

**Braided monoidal categories.** A monoidal category  $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$  consists of a category  $\mathcal{C}$ , a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , called the tensor product, an object  $I \in \mathcal{C}$  called the unit object, and natural isomorphisms  $a : \otimes \circ (\otimes \times \mathcal{C}) \rightarrow \otimes \circ (\mathcal{C} \times \otimes)$  (the associativity constraint),  $l : \otimes \circ (I \times \mathcal{C}) \rightarrow \mathcal{C}$  (the left unit constraint) and  $r : \otimes \circ (\mathcal{C} \times I) \rightarrow \mathcal{C}$  (the right unit constraint).  $a$ ,  $l$  and  $r$  have to satisfy certain coherence conditions, we refer to [10, XI.2] for a detailed discussion.  $\mathcal{C}$  is called strict if  $a$ ,  $l$  and  $r$  are the identities on  $\mathcal{C}$ . McLane's coherence Theorem asserts that every monoidal category is monoidal equivalent to a strict one, see [10, XI.5]. The categories that we will consider are - technically spoken - not strict, but they can be treated as if they were strict.

Let  $\tau : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  be the flip functor. A prebraiding on  $\mathcal{C}$  is a natural transformation  $c : \otimes \rightarrow \otimes \circ \tau$  satisfying the following equations, for all  $U, V, W \in \mathcal{C}$ :

$$c_{U, V \otimes W} = (V \otimes c_{U, W}) \circ (c_{U, V} \otimes W) ; \quad c_{U \otimes V, W} = (c_{U, W} \otimes V) \circ (U \otimes c_{V, W}).$$

$c$  is called a braiding if it is a natural isomorphism.  $c$  is called a symmetry if  $c_{U, V}^{-1} = c_{V, U}$ , for all  $U, V \in \mathcal{C}$ . We refer to [10, XIII.1] for more detail.

## 2. BRAIDINGS ON THE CATEGORY OF BIMODULES

Let  $A$  be an algebra over a commutative ring  $k$  and  ${}_A\mathcal{M}_A = ({}_A\mathcal{M}_A, - \otimes_A -, A)$  the monoidal category of  $A$ -bimodules.  $A^{(n)}$  will be a shorter notation for the  $n$ -fold tensor product  $A \otimes \cdots \otimes A$ , where  $\otimes = \otimes_k$ . An element  $R \in A^{(3)}$  will be denoted by  $R = R^1 \otimes R^2 \otimes R^3$ , where summation is implicitly understood. Our first aim is to investigate braidings on  ${}_A\mathcal{M}_A$ .

**Theorem 2.1.** *Let  $A$  be a  $k$ -algebra. Then there is a bijective correspondence between the class of all braidings  $c$  on  ${}_A\mathcal{M}_A$  and the set of all invertible elements  $R = R^1 \otimes R^2 \otimes R^3 \in A^{(3)}$  satisfying the following conditions, for all  $a \in A$ :*

$$\begin{aligned} (2) \quad & R^1 \otimes R^2 \otimes aR^3 = R^1 a \otimes R^2 \otimes R^3 \\ (3) \quad & aR^1 \otimes R^2 \otimes R^3 = R^1 \otimes R^2 a \otimes R^3 \\ (4) \quad & R^1 \otimes aR^2 \otimes R^3 = R^1 \otimes R^2 \otimes R^3 a \\ (5) \quad & R^1 \otimes R^2 \otimes 1 \otimes R^3 = r^1 R^1 \otimes r^2 \otimes r^3 R^2 \otimes R^3 \\ (6) \quad & R^1 \otimes 1 \otimes R^2 \otimes R^3 = R^1 \otimes R^2 r^1 \otimes r^2 \otimes R^3 r^3 \end{aligned}$$

where  $r = r^1 \otimes r^2 \otimes r^3 = R$ . Under the above correspondence the braiding  $c$  corresponding to  $R$  is given by the formula

$$(7) \quad c_{M,N} : M \otimes_A N \rightarrow N \otimes_A M, \quad c_{M,N}(m \otimes_A n) = R^1 n R^2 \otimes_A m R^3.$$

for all  $M, N \in {}_A\mathcal{M}_A$ ,  $m \in M$  and  $n \in N$ .

An invertible element  $R \in A^{(3)}$  satisfying (2)-(6) is called a canonical  $R$ -matrix and  $(A, R)$  is called an algebra with a canonical  $R$ -matrix.

*Proof.*  $A^{(2)}$  is an  $A$ -bimodule via the usual actions  $a(x \otimes y)b = ax \otimes yb$ , for all  $a, b, x, y \in A$ . Let  $c : {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A \rightarrow {}_A\mathcal{M}_A \times {}_A\mathcal{M}_A$  be a braiding on  ${}_A\mathcal{M}_A$ . For each  $M, N \in {}_A\mathcal{M}_A$ , we have an  $A$ -bimodule isomorphism  $c_{M,N} : M \otimes_A N \rightarrow N \otimes_A M$ , that is natural in  $M$  and  $N$ . Now consider

$$c_{A^{(2)}, A^{(2)}} : A^{(3)} \cong A^{(2)} \otimes_A A^{(2)} \rightarrow A^{(3)} \cong A^{(2)} \otimes_A A^{(2)},$$

and let  $R = c_{A^{(2)}, A^{(2)}}(1 \otimes 1 \otimes 1)$ .  $c$  is completely determined by  $R$ . For  $M, N \in {}_A\mathcal{M}_A$ ,  $m \in M$  and  $n \in N$ , we consider the  $A$ -bimodule maps  $f_m : A^{(2)} \rightarrow M$  and  $g_n : A^{(2)} \rightarrow N$  given by the formulas  $f_m(a \otimes b) = amb$  and  $g_n(a \otimes b) = anb$ . From the naturality of  $c$ , it follows that

$$(g_n \otimes_A f_m) \circ c_{A^{(2)}, A^{(2)}} = c_{M,N} \circ (f_m \otimes_A g_n).$$

(7) follows after we evaluate this formula at  $1 \otimes 1 \otimes 1$ . Obviously  $c_{M,N}(ma \otimes_A n) = c_{M,N}(m \otimes_A an)$ . Furthermore,  $c_{M,N}(am \otimes_A n) = ac_{M,N}(a \otimes_A n)$  and  $c_{M,N}(m \otimes_A na) = c_{M,N}(a \otimes_A n)a$  since  $c_{M,N}$  is a bimodule map. If we write these three formulas down in the case where  $M = N = A^{(2)}$ , and  $m = n = 1 \otimes 1$ , then we obtain (2-4).  $c$  satisfies the two triangle equalities

$$\begin{aligned} c_{M \otimes_A N, P} &= (c_{M,P} \otimes_A N) \circ (M \otimes_A c_{N,P}); \\ c_{M, N \otimes_A P} &= (N \otimes_A c_{M,P}) \circ (c_{M,N} \otimes_A P). \end{aligned}$$

The first equality is equivalent to

$$R^1 p R^2 \otimes_A m \otimes_A n R^3 = r^1 R^1 p R^2 r^2 \otimes_A m r^3 \otimes_A n R^3,$$

for all  $m \in M$ ,  $n \in N$  and  $p \in P$ . If we take  $M = N = P = A^{(2)}$  and  $m = n = p = 1 \otimes 1$ , then we find that  $R^1 \otimes R^2 \otimes 1 \otimes R^3 = r^1 R^1 \otimes R^2 r^2 \otimes r^3 \otimes R^3$ . Applying (4), we find that (5) holds. In a similar way, the second triangle equality implies (6).

We can apply the same arguments to the inverse braiding  $c^{-1}$ . This gives  $S = S^1 \otimes S^2 \otimes S^3 = c_{A^{(2)}, A^{(2)}}^{-1}(1 \otimes 1 \otimes 1)$ . Then we have that

$$m \otimes_A n = (c_{N,M}^{-1} \circ c_{M,N})(m \otimes_A n) = c_{N,M}^{-1}(R^1 n R^2 \otimes_A m R^3) = S^1 m R^3 S^2 \otimes_A R^1 n R^2 S^3.$$

Now take  $m = n = 1 \otimes 1 \in A^{(2)}$ . Then we find

$$\begin{aligned} 1 \otimes 1 \otimes 1 &= S^1 \otimes R^3 S^2 R^1 \otimes R^2 S^3 \stackrel{(3)}{=} R^1 S^1 \otimes R^3 S^2 \otimes R^2 S^3 \\ &\stackrel{(4)}{=} R^1 S^1 \otimes S^2 \otimes R^2 S^3 R^3 \stackrel{(2)}{=} R^1 S^1 R^2 \otimes S^2 \otimes S^3 R^3 \stackrel{(3)}{=} S^1 R^2 \otimes S^2 R^1 \otimes S^3 R^3. \end{aligned}$$

In a similar way, we have that  $R^1 S^2 \otimes R^2 S^1 \otimes R^3 S^3 = 1 \otimes 1 \otimes 1$ , and it follows that  $S^2 \otimes S^1 \otimes S^3$  is the inverse of  $R^1 \otimes R^2 \otimes R^3$ .

Conversely, assume that  $R \in A^{(3)}$  is invertible and satisfies (2-6). Then we define  $c$  using (7). Straightforward computations show that  $c$  is a braiding on  ${}_A \mathcal{M}_A$ .  $\square$

Let  $c$  be a braiding on  ${}_A \mathcal{M}_A$  and  $R$  the corresponding canonical  $R$ -matrix. Then  $c$  is a symmetry, if and only if  $S = R$ , this means that

$$(8) \quad R^2 r^1 \otimes R^1 r^2 \otimes R^3 r^3 = 1 \otimes 1 \otimes 1$$

that is,  $R^{-1} = R^2 \otimes R^1 \otimes R^3$ . The next theorem shows that the list of equations satisfied by an  $R$ -matrix from Theorem 2.1 can be reduced to two equations and furthermore, we prove that all braidings on the category of  $A$ -bimodules are symmetries.

**Theorem 2.2.** *Let  $A$  be a  $k$ -algebra. Then there is a bijection between the set of canonical  $R$ -matrices and the set of all invertible elements  $R \in A^{(3)}$  satisfying (4) and the normalizing condition*

$$(9) \quad R^1 R^2 \otimes R^3 = R^2 \otimes R^3 R^1 = 1 \otimes 1$$

*Furthermore, in this situation,  $R$  is invariant under cyclic permutation of the tensor factors,*

$$(10) \quad R = R^2 \otimes R^3 \otimes R^1 = R^3 \otimes R^1 \otimes R^2,$$

*and we have the additional normalizing condition*

$$(11) \quad R^1 \otimes R^2 R^3 = 1 \otimes 1.$$

*In particular, every braiding on  ${}_A \mathcal{M}_A$  is a symmetry.*

*Proof.* Let  $R$  be an  $R$ -matrix as in Theorem 2.1, i.e.  $R$  is invertible and satisfies (2-6). Multiplying the second and the third tensor factor in (6), we find that  $R = R^1 \otimes R^2 r^1 r^2 \otimes R^3 r^3 = R(1 \otimes r^1 r^2 \otimes r^3)$ . From the fact that  $R$  is invertible, it follows that  $1 \otimes 1 \otimes 1 = 1 \otimes r^1 r^2 \otimes r^3$ , and the first normalizing condition of (9) follows after we multiply the first two tensor factors. On the other hand, if we apply the flip map on

the last two positions in (5) we obtain that  $R^1 \otimes R^2 \otimes R^3 \otimes 1 = r^1 R^1 \otimes r^2 \otimes R^3 \otimes r^3 R^2$ . Multiplying the last two positions we obtain:

$$R = r^1 R^1 \otimes r^2 \otimes R^3 r^3 R^2 \stackrel{(2)}{=} r^1 R^3 R^1 \otimes r^2 \otimes r^3 R^2 = R(R^3 R^1 \otimes 1 \otimes R^2)$$

As  $R$  is invertible it follows that  $R^3 R^1 \otimes R^2 = 1 \otimes 1$ , as needed.

Conversely, assume now that  $R$  satisfies (4) and (9). We will show that  $R$  is a canonical  $R$ -matrix satisfying (8) and hence from the observations preceding Theorem 2.2 we obtain that the braiding  $c$  corresponding to  $R$  is a symmetry. First we show that  $R$  is invariant under cyclic permutation of the tensor factors.

$$\begin{aligned} R^3 \otimes R^1 \otimes R^2 &\stackrel{(9)}{=} R^3 r^1 r^2 \otimes r^3 R^1 \otimes R^2 \stackrel{(4)}{=} R^3 r^2 \otimes r^3 R^1 \otimes r^1 R^2 \\ &\stackrel{(4)}{=} r^2 \otimes r^3 R^3 R^1 \otimes r^1 R^2 \stackrel{(9)}{=} r^2 \otimes r^3 \otimes r^1. \end{aligned}$$

This implies immediately that the central conditions (2-3) are also satisfied. Next we show that (5-6) are satisfied.

$$\begin{aligned} r^1 R^1 \otimes r^2 \otimes r^3 R^2 \otimes R^3 &\stackrel{(3)}{=} R^1 \otimes r^2 \otimes r^3 R^2 r^1 \otimes R^3 \\ &\stackrel{(4)}{=} R^1 \otimes R^2 r^2 \otimes r^3 r^1 \otimes R^3 \stackrel{(9)}{=} R^1 \otimes R^2 \otimes 1 \otimes R^3; \\ R^1 \otimes R^2 r^1 \otimes r^2 \otimes R^3 r^3 &\stackrel{(4)}{=} R^1 \otimes r^3 R^2 r^1 \otimes r^2 \otimes R^3 \\ &\stackrel{(4)}{=} R^1 \otimes r^3 r^1 \otimes R^2 r^2 \otimes R^3 \stackrel{(9)}{=} R^1 \otimes 1 \otimes R^2 \otimes R^3. \end{aligned}$$

Finally, we prove that (8) holds since

$$\begin{aligned} R^1 r^2 \otimes R^2 r^1 \otimes R^3 r^3 &\stackrel{(3)}{=} R^1 r^2 R^2 \otimes r^1 \otimes R^3 r^3 \stackrel{(4)}{=} R^1 R^2 \otimes r^1 \otimes R^3 r^2 r^3 \\ &\stackrel{(9)}{=} 1 \otimes r^1 \otimes r^2 r^3 \stackrel{(10)}{=} 1 \otimes r^3 \otimes r^1 r^2 \stackrel{(9)}{=} 1 \otimes 1 \otimes 1. \end{aligned}$$

and the proof is finished.  $\square$

The commutative case is completely classified by the following result.

**Proposition 2.3.** *Let  $A$  be a  $k$ -algebra. Then:*

- (1) *If a monomial  $x \otimes y \otimes z$  is a canonical  $R$ -matrix, then it is equal to  $1 \otimes 1 \otimes 1$ .*
- (2)  *$1 \otimes 1 \otimes 1$  is a canonical  $R$ -matrix if and only if  $u_A : k \rightarrow A$  is an epimorphism of rings.*
- (3) *If  $A$  is commutative, then  $(A, R)$  is an algebra with a canonical  $R$ -matrix if and only if  $R = 1 \otimes 1 \otimes 1$  and  $u_A : k \rightarrow A$  is an epimorphism in the category of rings.*

*Proof.* 1. Let  $R = x \otimes y \otimes z$  be a canonical  $R$ -matrix. From (5-6), it follows that

$$x \otimes 1 \otimes y \otimes z = x \otimes yx \otimes y \otimes z^2 \text{ and } x \otimes y \otimes 1 \otimes z = x^2 \otimes y \otimes zy \otimes z.$$

Since  $R$  is invertible, this implies that

$$1 \otimes 1 \otimes 1 \otimes 1 = 1 \otimes yx \otimes 1 \otimes z \text{ and } 1 \otimes 1 \otimes 1 \otimes 1 = x \otimes 1 \otimes zy \otimes 1,$$

and, multiplying tensor factors, we find that  $1 \otimes 1 = yx \otimes z$  and  $1 \otimes 1 = x \otimes zy$ . It then follows that  $yxz = xzy = 1$ , hence  $y$  is invertible with  $y^{-1} = xz$ . Finally

$$x \otimes y \otimes z = x \otimes y^2 y^{-1} \otimes z \stackrel{(4)}{=} x \otimes yy^{-1} \otimes zy = 1 \otimes 1 \otimes 1.$$

2. If  $R = 1 \otimes 1 \otimes 1$ , then the three centralizing conditions (2-4) are equivalent to  $a \otimes 1 = 1 \otimes a$ , for all  $a \in A$ , which is equivalent to  $u_A : k \rightarrow A$  being an epimorphism

of rings, see [15].

3. Assume that  $(A, R)$  is an algebra with a canonical R-matrix. Then:

$$\begin{aligned} R^1 \otimes R^2 \otimes 1 \otimes R^3 &\stackrel{(5)}{=} r^1 R^1 \otimes r^2 \otimes r^3 R^2 \otimes R^3 \\ &\stackrel{(4)}{=} r^1 R^1 \otimes R^2 r^2 \otimes r^3 \otimes R^3 = \sum R^1 r^1 \otimes R^2 r^2 \otimes r^3 \otimes R^3. \end{aligned}$$

At the third step, we used the fact that  $A$  is commutative. From the fact that  $R$  is invertible, it follows that  $R^1 \otimes R^2 \otimes 1 \otimes R^3 = 1 \otimes 1 \otimes 1 \otimes 1$  and  $R = 1 \otimes 1 \otimes 1$ . The rest of the proof follows from 2.  $\square$

**Remarks 2.4.** 1. The notion of quasi-triangular bialgebroid was introduced in [7, Def. 19]. Quasi-triangular structures on a bialgebroid are given by universal R-matrices, see [7, Prop. 20] and [3, Def. 3.15], and correspond bijectively to braidings on the category of modules over the bialgebroid [3, Theorem 3.16]. It is well-known that  $A^e$  is an  $A$ -bialgebroid, with the Sweedler canonical coring as underlying coring, and  $A$ -bimodules are left  $A^e$ -modules. Comparing our Theorem 2.1 with the (left handed) version of [3, Theorem 3.16] yields that canonical R-matrices for  $A$  correspond bijectively to universal R-matrices for the canonical bialgebroid  $A^e$ . This leads to an alternative proof of Theorem 2.1, if we identify the R-matrices from [7] with our R-matrices. This, however, is more complicated than the straightforward proof that we presented, that also has the advantage that it is self-contained and avoids all technicalities on bialgebroids.

2. (5-6) can be rewritten as  $R^{124} = R^{123}R^{134}$  and  $R^{134} = R^{124}R^{234}$  in the algebra  $A^{(4)}$ .

3. It follows from Proposition 2.3 that there is only one braiding on the category of (left)  $k$ -modules, namely the one given by the usual switch map.

Before we state our next main result Theorem 2.6, we need a technical Lemma. If  $M \in {}_A\mathcal{M}_A$ , then  $A \otimes M$  is a  $k \otimes A$ -bimodule, and we can consider

$$(A \otimes M)^{k \otimes A} = \left\{ \sum_i a_i \otimes m_i \in A \otimes M \mid \sum_i a_i \otimes am_i = \sum_i a_i \otimes m_i a, \text{ for all } a \in A \right\}.$$

If  $M = A^{(2)}$ , then  $(A \otimes A^{(2)})^{k \otimes A}$  is the set of elements  $R \in A^{(3)}$  satisfying (4). We have a map  $\alpha_M : A \otimes M^A \rightarrow (A \otimes M)^A$ ,  $\alpha_M(a \otimes m) = a \otimes m$ .

**Lemma 2.5.** *Let  $M$  be an  $A$ -bimodule. The map  $\alpha_M$  is injective if  $A$  is flat as a  $k$ -module, and bijective if  $A$  is free as a  $k$ -module.*

*Proof.* If  $A$  is flat, then  $A \otimes M^A \rightarrow A \otimes M$  is injective, and then  $\alpha_M$  is also injective. Assume that  $A$  is free as a  $k$ -module, and let  $\{e_j \mid j \in I\}$  be a free basis of  $A$ . Assume that  $x = \sum_i a_i \otimes m_i \in (A \otimes M)^{k \otimes A}$ . For all  $i$ , we can write  $a_i = \sum_{j \in I} \alpha_i^j e_j$ , for some  $\alpha_i^j \in k$ . Then  $x = \sum_{j \in I} e_j \otimes (\sum_i \alpha_i^j m_i)$ . Now

$$x = \sum_{j \in I} e_j \otimes \left( \sum_i \alpha_i^j am_i \right) = \sum_i a_i \otimes am_i = \sum_i a_i \otimes m_i a = \sum_{j \in I} e_j \otimes \left( \sum_i \alpha_i^j m_i a \right),$$

hence  $\sum_i \alpha_i^j am_i = \sum_i \alpha_i^j m_i a$ , for all  $j \in I$ , and  $\sum_i \alpha_i^j m_i \in M^A$ . We conclude that  $x = \sum_{j \in I} e_j \otimes (\sum_i \alpha_i^j m_i) \in \text{Im } \alpha_M$ , and this shows that  $\alpha_M$  is surjective.  $\square$

**Theorem 2.6.** *Let  $A$  be a  $k$ -algebra  $A$ , and consider the conditions:*



- (1)  $(F, G)$  is a pair of inverse equivalences, that is,  $A$  is an Azumaya algebra;
- (2) The functor  $G = (-)^A : {}_A\mathcal{M}_A \rightarrow \mathcal{M}_k$  is fully faithful;
- (3) the functor  $G = (-)^A : {}_A\mathcal{M}_A \rightarrow \mathcal{M}_k$  is separable;
- (4) there exists  $R = R^1 \otimes R^2 \otimes R^3 \in A \otimes (A \otimes A)^A$  such that  $R^1 R^2 \otimes R^3 = 1 \otimes 1$ ;
- (5) there exists a unique  $R = R^1 \otimes R^2 \otimes R^3 \in A \otimes (A \otimes A)^A$  such that  $R^1 R^2 \otimes R^3 = 1 \otimes 1$ ;
- (6) there exists a braiding on  ${}_A\mathcal{M}_A$ , that is, there exists  $R \in A^{(3)}$  such that  $(A, R)$  is an algebra with a canonical  $R$ -matrix.

Then  $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Rightarrow (6)$ . If  $A$  is central, then  $(2) \Rightarrow (1)$ . If  $A$  is free as a  $k$ -module, then  $(6) \Rightarrow (5)$ , and in this case the braiding on  ${}_A\mathcal{M}_A$  is unique. If  $k$  is a field, and  $A$  is finite dimensional, then  $(6) \Rightarrow (1)$ , and all six assertions are equivalent.

*Proof.*  $(1) \Rightarrow (2)$ ,  $(2) \Rightarrow (3)$  and  $(5) \Rightarrow (4)$  are trivial.

$(3) \Rightarrow (4)$ . If  $G$  is separable, then we have a natural transformation  $\zeta : 1 \Rightarrow FG$  such that  $\varepsilon_M \circ \zeta_M = M$ , for all  $M \in {}_A\mathcal{M}_A$ . Now let  $R = \zeta_{A^{(2)}}(1 \otimes 1) = R^1 \otimes R^2 \otimes R^3 \in FG(A^{(2)}) = A \otimes (A \otimes A)^A$ . Then  $1 \otimes 1 = (\varepsilon_{A^{(2)}} \circ \zeta_{A^{(2)}})(1 \otimes 1) = R^1 R^2 \otimes R^3$ .

The natural transformation  $\zeta$  is completely determined by  $R$ . For an  $A$ -bimodule  $M$  and  $m \in M$ , we define  $f_m$  as in the proof of Theorem 2.1. From the naturality of  $\zeta$ , it follows that the diagram

$$\begin{array}{ccc} A^{(2)} & \xrightarrow{\zeta_{A^{(2)}}} & A \otimes (A \otimes A)^A \\ f_m \downarrow & & \downarrow A \otimes (f_m)^A \\ M & \xrightarrow{\zeta_M} & A \otimes M^A \end{array}$$

commutes. Evaluating the diagram at  $1 \otimes 1$ , we find that

$$(12) \quad \zeta_M(m) = R^1 \otimes R^2 m R^3.$$

$(4) \Rightarrow (6)$ . Write  $R = \sum_i a_i \otimes b_i$ , with  $a_i \in A$  and  $b_i \in (A \otimes A)^A$ . Then  $R^2 \otimes R^3 R^1 = \sum_i b_i a_i = \sum_i a_i b_i = R^1 R^2 \otimes R^3 = 1 \otimes 1$ , hence  $\alpha_{A^{(2)}}(R) \in (A \otimes A^{(2)})^{k \otimes A}$  satisfies (4), and it follows from Theorem 2.2 that  $\alpha_{A^{(2)}}(R)$  determines a braiding on  ${}_A\mathcal{M}_A$ . It also follows from Theorem 2.2 that (3) and (11) are satisfied.

$(4) \Rightarrow (2)$ . Given  $R \in A \otimes (A \otimes A)^A$  satisfying  $R^1 R^2 \otimes R^3 = 1 \otimes 1$ , we define  $\zeta$  using (12). It follows immediately that  $(\varepsilon_M \circ \zeta_M)(m) = \varepsilon(R^1 \otimes R^2 m R^3) = R^1 R^2 m R^3 = m$ .

We have seen in the proof of  $(4) \Rightarrow (6)$  that (3) and (11) are satisfied. For  $a_i \in A$  and  $m_i \in M^A$ , we then compute

$$\begin{aligned} (\zeta_M \circ \varepsilon_M)(\sum_i a_i \otimes m_i) &= \sum_i R^1 \otimes R^2 a_i m_i R^3 \stackrel{(3)}{=} \sum_i a_i R^1 \otimes R^2 m_i R^3 \\ &= \sum_i a_i R^1 \otimes R^2 R^3 m_i \stackrel{(11)}{=} \sum_i a_i \otimes m_i. \end{aligned}$$

This shows that  $\varepsilon$  is a natural transformation with inverse  $\zeta$ , and  $G$  is fully faithful.

$(2) \Rightarrow (5)$ . We have already seen that 2) implies 4), and this shows that  $R$  exists. If  $G$  is fully faithful, then  $\varepsilon_M$  is invertible, for all  $M \in {}_A\mathcal{M}_A$ . If  $R \in A \otimes (A \otimes A)^A$  satisfies  $R^1 R^2 \otimes R^3 = 1 \otimes 1$ , then  $\varepsilon_{A \otimes A}(R) = 1 \otimes 1$ , hence  $R = \varepsilon_{A \otimes A}^{-1}(1 \otimes 1)$ .



(6)  $\Rightarrow$  (4). From (5), it follows that there exists  $R \in (A \otimes A^{(2)})^{k \otimes A}$  such that  $R^1 R^2 \otimes R^3 = 1 \otimes 1$ , see Theorem 2.2.  $\alpha_{A^{(2)}}$  is bijective, see Lemma 2.5, hence  $\alpha_{A^{(2)}}^{-1}(R) \in A \otimes (A \otimes A)^A$  satisfies (3). The uniqueness of  $R$  follows from (4).

(4)  $\Rightarrow$  (1). Assume that  $A$  is central. From (4), it follows that  $\varepsilon_{A \otimes A} : A \otimes (A \otimes A)^A \rightarrow A \otimes A$  is surjective, and then it follows from [1, Theorem 3.1] that  $A$  is separable over  $Z(A) = k$ . Thus  $A$  is central separable, and therefore Azumaya.

(6)  $\Rightarrow$  (1). If  $k$  is a field, then  $A$  is free, so (6) implies (5), and, a fortiori, (2). Then  $\varepsilon_A : A \otimes A^A \rightarrow A$  is an isomorphism of  $A$ -bimodules, and therefore also of vector spaces. A count of dimensions shows that  $\dim_k(Z(A)) = \dim_k(A^A) = 1$ , so that  $Z(A) = k1_A$ , and  $A$  is central, and then (1) follows from (2).  $\square$

In particular, applying Theorem 2.6 for finite dimensional algebras over fields we obtain the following surprising characterization of central simple algebras:

**Corollary 2.7.** *For a finite dimensional algebra  $A$  over a field  $k$ , the following assertions are equivalent:*

- (1)  $A$  is a central simple algebra;
- (2) there exists a (unique) braiding on  ${}_A \mathcal{M}_A$ ;
- (3) there exists a (unique) invertible element  $R \in A \otimes A \otimes A$  satisfying the conditions  $R^1 \otimes a R^2 \otimes R^3 = R^1 \otimes R^2 \otimes R^3 a$  and  $R^1 R^2 \otimes R^3 = R^2 \otimes R^3 R^1 = 1 \otimes 1$ , for all  $a \in A$ .

For any  $k$ -algebra  $A$ , the functor  $F : \mathcal{M}_k \rightarrow {}_A \mathcal{M}_A$  is strong monoidal. Indeed, for any  $N, N' \in \mathcal{M}_k$ , we have natural isomorphisms  $\varphi_0 : F(k) = A \otimes k \rightarrow A$  and

$$\varphi_{N, N'} : F(N) \otimes_A F(N') = (A \otimes N) \otimes_A (A \otimes N') \rightarrow F(N \otimes N') = A \otimes N \otimes N'$$

satisfying all the necessary axioms, see [10].

**Proposition 2.8.** *Let  $(A, R)$  be an algebra with a canonical  $R$ -matrix. Then the symmetry on  ${}_A \mathcal{M}_A$  is such that the functor  $F : \mathcal{M}_k \rightarrow {}_A \mathcal{M}_A$  preserves the symmetry.*

*Proof.* We have to show that the following diagram commutes

$$\begin{array}{ccc} (A \otimes N) \otimes_A (A \otimes N') & \xrightarrow{\varphi_{N, N'}} & A \otimes N \otimes N' \\ \downarrow c_{A \otimes N, A \otimes N'} & & \downarrow A \otimes \tau_{N, N'} \\ (A \otimes N') \otimes_A (A \otimes N) & \xrightarrow{\varphi_{N', N}} & A \otimes N' \otimes N \end{array}$$

Here  $\tau_{N, N'} : N \otimes N' \rightarrow N' \otimes N$  is the usual switch map. For  $a, b \in A$ ,  $n \in N$  and  $n' \in N'$ , we compute

$$\begin{aligned} & (\varphi_{N', N} \circ c_{A \otimes N, A \otimes N'})((a \otimes n) \otimes_A (b \otimes n')) \stackrel{(7)}{=} \varphi_{N', N}((R^1 b R^2 \otimes n') \otimes_A a R^3 \otimes n) \\ & \quad = R^1 b R^2 a R^3 \otimes n' \otimes n \stackrel{(2)}{=} R^1 R^2 a b R^3 \otimes n' \otimes n \\ & \stackrel{(9)}{=} a b \otimes n' \otimes n = a b \otimes \tau_{N, N'}(n \otimes n') \\ & \quad = ((A \otimes \tau_{N, N'}) \circ \varphi_{N, N'})((a \otimes n) \otimes_A (b \otimes n')) \end{aligned}$$

and the proof is finished.  $\square$

If  $A$  is an Azumaya algebra, then it follows from Theorem 2.6 that we have a symmetry on the category of  $A$ -bimodules  ${}_A\mathcal{M}_A$ . In Examples 2.9 and 2.10, we give explicit formulas for  $R$  in the case where  $A$  is a matrix ring or a quaternion algebra; in both cases  $A$  is free, so that the canonical  $R$ -matrix is unique.

**Example 2.9.** Let  $A = M_n(k)$  be a matrix algebra. Then the  $R$ -matrix for  $M_n(k)$  is given by

$$R = \sum_{i,j,k=1}^n e_{ij} \otimes e_{ki} \otimes e_{jk}$$

where  $e_{ij}$  is the elementary matrix with 1 in the  $(i, j)$ -position and 0 elsewhere. Indeed, for all indices  $i, j, p, q$ , we have

$$e_{pq} \left( \sum_{k=1}^n e_{ki} \otimes e_{jk} \right) = e_{pi} \otimes e_{jq} = \left( \sum_{k=1}^n e_{ki} \otimes e_{jk} \right) e_{pq},$$

hence  $\sum_{k=1}^n e_{ki} \otimes e_{jk} \in (A \otimes A)^A$  and  $R = \sum_{i,j=1}^n e_{ij} \otimes \left( \sum_{k=1}^n e_{ki} \otimes e_{jk} \right) \in A \otimes (A \otimes A)^A$ . Finally

$$\sum_{i,j,k=1}^n e_{ij} e_{ki} \otimes e_{jk} = \sum_{i,j=1}^n e_{ii} \otimes e_{jj} = 1 \otimes 1$$

as needed.

**Example 2.10.** Let  $K$  be a commutative ring, such that 2 is invertible in  $K$ , and take two invertible elements  $a, b \in K$ . The generalized quaternion algebra  $A = {}^aK^b$  is the free  $K$ -module with basis  $\{1, i, j, k\}$  and multiplication defined by

$$i^2 = a, \quad j^2 = b, \quad ij = -ji = k.$$

It is well-known that  $A$  is an Azumaya algebra. The corresponding  $R$ -matrix is

$$\begin{aligned} R = & \frac{1}{4}(1 \otimes 1 \otimes 1) + \frac{1}{4a}(1 \otimes i \otimes i + i \otimes 1 \otimes i + i \otimes i \otimes 1) \\ & + \frac{1}{4b}(1 \otimes j \otimes j + j \otimes 1 \otimes j + j \otimes j \otimes 1) - \frac{1}{4ab}(1 \otimes k \otimes k + k \otimes 1 \otimes k + k \otimes k \otimes 1) \\ & + \frac{1}{4ab}(i \otimes j \otimes k + j \otimes k \otimes i + k \otimes i \otimes j) - \frac{1}{4ab}(j \otimes i \otimes k + k \otimes j \otimes i + i \otimes k \otimes j). \end{aligned}$$

It is easy to show that  $R$  satisfies (4) and (9). Indeed,

$$\begin{aligned} R^1 R^2 \otimes R^3 = & \frac{1}{4}(1 \otimes 1) + \frac{1}{4a}(i \otimes i + i \otimes i + a \otimes 1) \\ & + \frac{1}{4b}(j \otimes j + j \otimes j + b \otimes 1) - \frac{1}{4ab}(k \otimes k + k \otimes k - ab \otimes 1) \\ & + \frac{1}{4ab}(k \otimes k - bi \otimes i - aj \otimes j) + \frac{1}{4ab}(k \otimes k - bi \otimes i - aj \otimes j) = 1 \otimes 1, \end{aligned}$$

proving the first normalization from (9); the second one follows in a similar manner. An elementary computation shows that  $R^1 \otimes x R^2 \otimes R^3 = R^1 \otimes R^2 \otimes R^3 x$ , for  $x = i, j, k$ , and this proves (4).

**Example 2.11.** Any Azumaya algebra  $A$  admits a canonical R-matrix. The converse is not true: it suffices to consider  $\mathbb{Q}$  as a  $\mathbb{Z}$ -algebra. Since  $\mathbb{Z} \subset \mathbb{Q}$  is an epimorphism of rings, it follows from Proposition 2.3 that  $(\mathbb{Q}, 1 \otimes 1 \otimes 1)$  is braided and it is obvious that  $\mathbb{Q}$  is not a  $\mathbb{Z}$ -Azumaya algebra.

**Example 2.12.** Let  $(A, R)$ ,  $(B, S)$  be two algebras with a canonical R-matrix. It is straightforward to show that  $(A \otimes B, T)$ , with  $T := R^1 \otimes S^1 \otimes R^2 \otimes S^2 \otimes R^3 \otimes S^3 \in (A \otimes B)^{(3)}$ , is an algebra with a canonical R-matrix.

We conclude this paper with another application of canonical R-matrices: they can be applied to deform the switch map into a simultaneous solution of the quantum Yang-Baxter equation and the braid equation.

**Theorem 2.13.** *Let  $(A, R)$  be an algebra with a canonical R-matrix and  $V$  an  $A$ -bimodule. Then the map*

$$\Omega : V \otimes V \rightarrow V \otimes V, \quad \Omega(v \otimes w) = R^1 w R^2 \otimes R^3 v$$

*is a solution of the quantum Yang-Baxter equation  $\Omega^{12} \Omega^{13} \Omega^{23} = \Omega^{23} \Omega^{13} \Omega^{12}$  and of the braid equation  $\Omega^{12} \Omega^{23} \Omega^{12} = \Omega^{23} \Omega^{12} \Omega^{23}$ . Moreover  $\Omega^3 = \Omega$  in  $\text{End}(V^{(3)})$ .*

*Proof.* Let  $r = S = R$ . Then for any  $v, w, t \in V$  we have:

$$\begin{aligned} \Omega^{12} \Omega^{13} \Omega^{23}(v \otimes w \otimes t) &= \Omega^{12} \Omega^{13}(v \otimes R^1 t R^2 \otimes R^3 w) \\ &= \Omega^{12}(r^1 R^3 w r^2 \otimes R^1 t R^2 \otimes r^3 v) = S^1 R^1 t R^2 S^2 \otimes S^3 r^1 R^3 w r^2 \otimes r^3 v \\ &\stackrel{(3)}{=} R^1 t R^2 S^1 S^2 \otimes S^3 r^1 R^3 w r^2 \otimes r^3 v \stackrel{(9)}{=} R^1 t R^2 \otimes r^1 R^3 w r^2 \otimes r^3 v \\ &\stackrel{(2)}{=} R^1 t R^2 \otimes r^1 w r^2 \otimes R^3 r^3 v \end{aligned}$$

and

$$\begin{aligned} \Omega^{23} \Omega^{13} \Omega^{12}(v \otimes w \otimes t) &= \Omega^{23} \Omega^{13}(R^1 w R^2 \otimes R^3 v \otimes t) \\ &= \Omega^{23}(r^1 t r^2 \otimes R^3 v \otimes r^3 R^1 w R^2) = r^1 t r^2 \otimes S^1 r^3 R^1 w R^2 S^2 \otimes S^3 R^3 v \\ &\stackrel{(2)}{=} r^1 t r^2 \otimes S^1 R^1 w R^2 S^2 \otimes r^3 S^3 R^3 v \stackrel{(3)}{=} r^1 t r^2 \otimes R^1 w R^2 S^1 S^2 \otimes r^3 S^3 R^3 v \\ &\stackrel{(9)}{=} r^1 t r^2 \otimes R^1 w R^2 \otimes r^3 R^3 v, \end{aligned}$$

Hence  $\Omega$  is a solution of the quantum Yang-Baxter equation. On the other hand:

$$\begin{aligned} \Omega^{12} \Omega^{23} \Omega^{12}(v \otimes w \otimes t) &= \Omega^{12} \Omega^{23}(R^1 w R^2 \otimes R^3 v \otimes t) \\ &= \Omega^{12}(R^1 w R^2 \otimes r^1 t r^2 \otimes r^3 R^3 v) = S^1 r^1 t r^2 S^2 \otimes S^3 R^1 w R^2 \otimes r^3 R^3 v \\ &\stackrel{(3)}{=} r^1 t r^2 S^1 S^2 \otimes S^3 R^1 w R^2 \otimes r^3 R^3 v \stackrel{(9)}{=} r^1 t r^2 \otimes R^1 w R^2 \otimes r^3 R^3 v \end{aligned}$$

and

$$\begin{aligned} \Omega^{23} \Omega^{12} \Omega^{23}(v \otimes w \otimes t) &= \Omega^{23} \Omega^{12}(v \otimes r^1 t r^2 \otimes r^3 w) \\ &= \Omega^{23}(S^1 r^1 t r^2 S^2 \otimes S^3 v \otimes r^3 w) = S^1 r^1 t r^2 S^2 \otimes R^1 r^3 w R^2 \otimes R^3 S^3 v \\ &\stackrel{(3)}{=} r^1 t r^2 S^1 S^2 \otimes R^1 r^3 w R^2 \otimes R^3 S^3 v \stackrel{(9)}{=} r^1 t r^2 \otimes R^1 r^3 w R^2 \otimes R^3 v \\ &\stackrel{(2)}{=} r^1 t r^2 \otimes R^1 w R^2 \otimes r^3 R^3 v. \end{aligned}$$

Thus  $\Omega$  is also a solution of the braid equation. Finally,

$$\begin{aligned}
\Omega^3(v \otimes w) &= S^1 r^3 R^1 w R^2 S^2 \otimes S^3 r^1 R^3 v r^2 \\
&\stackrel{(2)}{=} S^1 w R^2 S^2 \otimes r^3 R^1 S^3 r^1 R^3 v r^2 \stackrel{(11)}{=} S^1 w S^2 \otimes r^3 R^1 S^3 R^2 r^1 R^3 v r^2 \\
&\stackrel{(3)}{=} S^1 w S^2 \otimes R^1 S^3 R^2 r^3 r^1 R^3 v r^2 \stackrel{(9)}{=} S^1 w S^2 \otimes R^1 S^3 R^2 R^3 v \\
&\stackrel{(10)}{=} S^1 w S^2 \otimes S^3 v = \Omega(v \otimes w).
\end{aligned}$$

□

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FACULTY OF ENGINEERING, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2, B-1050 BRUSSELS, BELGIUM  
*E-mail address:* ana.agore@vub.ac.be and ana.agore@gmail.com

FACULTY OF ENGINEERING, VRIJE UNIVERSITEIT BRUSSEL, PLEINLAAN 2, B-1050 BRUSSELS, BELGIUM  
*E-mail address:* scaenepe@vub.ac.be

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BUCHAREST, STR. ACADEMIEI 14, RO-010014 BUCHAREST 1, ROMANIA  
*E-mail address:* gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com